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NEW EXAMPLES OF LAGRANGIAN RIGIDITY

BY

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ABSTRACT

Let $n = 4$ or 8. We prove that any Lagrangian embedding of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ into $\mathbb{C}^n$ has a trivial linking class. We deduce that every embedding of $\mathbb{S}^3 \times \mathbb{S}^1$ into $\mathbb{C}^4$ is isotopic to a Lagrangian embedding. This is false if $n = 8$.

1. Introduction

Given a totally real embedding j of a compact, oriented, manifold M^n without boundary into $\mathbb{C}^n$ and a nowhere vanishing tangent vector field v, one can build a cohomology class $\sigma_{n-1}(j,v) \in H^{n-1}(M^n, \mathbb{Z})$ called the **linking class** of j associated to v. This class is an invariant of isotopy classes of totally real embeddings. If $j_1, j_2\colon M^n \longrightarrow \mathbb{C}^n$ are two totally real embeddings and if there exists a nonvanishing vector field v such that $\sigma_{n-1}(j_1,v) \neq \sigma_{n-1}(j_2,v)$, then j_1 and j_2 are not isotopic through totally real embeddings.

If M^n is the 2-torus and v the translation field on $\mathbb{T}^2$, it has been shown in [12] that for every $\sigma \in H^1(\mathbb{T}^2, \mathbb{Z})$ there exists a totally real embedding $j\colon \mathbb{T}^2 \longrightarrow \mathbb{C}^2$ such that $\sigma_1(j,v) = \sigma$. Nevertheless, if one looks for Lagrangian embeddings, then the situation is completely different. In 1994, Y. Eliashberg and L. Polterovich proved that for every Lagrangian embedding $j\colon \mathbb{T}^2 \longrightarrow \mathbb{C}^2$, one has: $\sigma_1(j,v) = 0$ ([5]).

Here we consider the product manifolds $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^7$. It is easily checked that these manifolds admit a Lagrangian embedding into $\mathbb{C}^n$, for n =4

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or 8. This follows from the Gromov–Lees theorem on Lagrangian immersions [10] combined with a result of M. Audin (see [1] pp. 617–618 and also remark B below). Moreover, it has been shown in [3] that these manifolds admit a family of totally real embeddings such that $\sigma_{n-1}(j,v)$ takes a countable number of different values. Using the techniques developed by Y. Eliashberg and L. Polterovich, we prove the following rigidity results.

THEOREM 1: *Let $n = 4$ or 8 and $j\colon \mathbb{S}^1\times\mathbb{S}^{n-1} \longrightarrow \mathbb{C}^n$ be a Lagrangian embedding. Let v be a nonvanishing vector field on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ tangent to the factor $\mathbb{S}^1$. Then $\sigma_{n-1}(j,v) = 0$.*

By a slight modification of the proof of Theorem 1, we also obtain similar results for some quotients of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $n = 4$ or 8 (see section 4).

Let $E(M^n, \mathbb{C}^n)$ be the space of embeddings of M^n into $\mathbb{C}^n$ and $E_{Lag}(M^n, \mathbb{C}^n)$ be the subspace of Lagrangian embeddings (all those spaces are endowed with the compact-open topology). We apply the preceding results to study the natural inclusion $i\colon E_{Lag}(M^n, \mathbb{C}^n) \longrightarrow E(M^n, \mathbb{C}^n)$ with $M^n = \mathbb{S}^1\times\mathbb{S}^{n-1}$ and $n = 4$ or 8.

THEOREM 2: *The map*

$$i_\sharp\colon \pi_0(E_{Lag}(\mathbb{S}^1\times\mathbb{S}^{n-1},\mathbb{C}^n)) \longrightarrow \pi_0(E(\mathbb{S}^1\times\mathbb{S}^{n-1},\mathbb{C}^n)) \simeq \mathbb{Z}_2$$

$$\textit{is} \begin{cases} \textit{onto} & \textit{if } n = 4, \\ \textit{a constant map} & \textit{if } n = 8. \end{cases}$$

The group $\mathrm{Diff}(M^n)$ acts on $E(M^n, \mathbb{C}^n)$ in an obvious way by reparametrizations of M^n. We shall see that $\mathrm{Diff}(\mathbb{S}^1\times\mathbb{S}^3)$ acts trivially on $\pi_0(E(\mathbb{S}^1\times\mathbb{S}^3,\mathbb{C}^4))$. Theorem 2 implies that there exist two Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C}^4$ such that the images of these maps are not isotopic in $\mathbb{C}^4$ as nonparametrized submanifolds.

Given two Lagrangian immersions $j_0, j_1\colon M^n \longrightarrow \mathbb{C}^n$ we define a cohomology class $\Delta(j_0,j_1) \in H^3(M^n,\mathbb{Z})$ which is an obstruction to the existence of a Lagrangian regular homotopy joining j_0 and j_1. If j_0 and j_1 are two Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^3$, it turns out that the modulo 2 reduction of this class — denoted by $\epsilon(j_0,j_1)$ — is the only obstruction to the existence of a smooth isotopy between j_0 and j_1. Of course, $\epsilon(j_0,j_1) \neq 0$ also implies that j_0 and j_1 belong to different components of the space $I_{Lag}(\mathbb{S}^1\times\mathbb{S}^3,\mathbb{C}^4)$ of Lagrangian immersions. In particular, two Lagrangian embeddings which correspond to different elements of $\pi_0(E(\mathbb{S}^1\times\mathbb{S}^3,\mathbb{C}^4))$ are not only nonisotopic as smooth embeddings but also not regular homotopic as Lagrangian immersions.

It is also worth noting that all Lagrangian immersions and, in particular, all

Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ into $\mathbb{C}^n$ (n =4 or 8) are regular homotopic as smooth immersions. This is due to M. Audin, see [1] theorem 0.1.

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2. Definitions and notations

Let M^n be a compact, connected, oriented, n-manifold without boundary. Let $< \cdot, \cdot >$ denote the Euclidean scalar product of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, J the canonical complex structure and $\omega =< J\cdot, \cdot >$ the symplectic form. An embedding $j\colon M^n \longrightarrow \mathbb{C}^n$ is called **totally real** if $T_{j(p)}j(M^n) \oplus JT_{j(p)}j(M^n) = \mathbb{C}^n$ for every $p \in M^n$. It is called **Lagrangian** if $JT_{j(p)}j(M^n) = T^{\perp}_{j(p)}j(M^n)$ for every $p \in M^n$, or equivalently if $j^*\omega \equiv 0$. Since $JTM^n \oplus TM^n = \mathbb{C}^n$, every manifold admitting a totally real embedding necessarily has a zero Euler–Poincaré characteristic. Thus, there exists a nowhere vanishing vector field v on M^n. One can stretch the submanifold $j(M^n)$ in the direction $Jdj(v)$ by a small distance ϵ. We denote by $Jv(M^n)$ the resulting submanifold. If ϵ is small enough, $Jv(M^n)$ is disjoint from $j(M^n)$. It represents a n-cycle in $\mathbb{C}^n \setminus j(M^n)$. Using the exact sequence of the pair $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ and Alexander duality, one obtains the following identifications:

$$H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H_{n+1}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

Definition (cf. [3]): The **linking class** associated to the totally real embedding j and to the vector field v is the class

$$\sigma_{n-1}(j, v) \in H^{n-1}(M^n, \mathbb{Z})$$

represented in $H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z})$ by the n-cycle $[Jv(M^n)]$.

3. Proof of Theorem 1

3.1 A MODEL FOR $\mathbb{S}^1 \times \mathbb{S}^3$ AND $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$. Let $\mathbb{H}$ denote the quaternionic field and let $x = x_0 + x_1 i + x_2 j + x_3 k$ be any element of $\mathbb{H}$. Consider the free action of $\mathbb{Z}$ on $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$, where the generator of $\mathbb{Z}$ acts by multiplication by 2. The quotient $\mathbb{H}^*/\mathbb{Z}$ can be identified with $\mathbb{S}^1 \times \mathbb{S}^3$. Let $<,>$ denote the canonical scalar product on $\mathbb{H}^*$ ($< x, x' >= \mathrm{Re}(x\overline{x'})$). A basis of $T^*_x(\mathbb{S}^1 \times \mathbb{S}^3)$ is given by

$$\theta_0 = \frac{1}{|x|^2} < x, \cdot >, \quad \theta_1 = \frac{1}{|x|^2} < xi, \cdot >,$$
$$\theta_2 = \frac{1}{|x|^2} < xj, \cdot >, \quad \theta_3 = \frac{1}{|x|^2} < xk, \cdot >,$$

and the Liouville form at a point (x, y) in $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$ will be given by

$$\lambda_{(x,y)} = y_0\theta_0 + y_1\theta_1 + y_2\theta_2 + y_3\theta_3 = \frac{1}{|x|^2} < xy, \cdot >= \frac{1}{|x|^2} \operatorname{Re}(xyd\overline{x}).$$

Let $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$ be the unit cotangent bundle of $\mathbb{S}^1 \times \mathbb{S}^3$, and let ξ be the restriction of the Liouville form on $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$. Then ξ is the canonical contact form of $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$. Later on, we shall need the following lemma.

Lemma A: *Let*

$$\Xi_1 = d\Big(\frac{1}{|x|^2}\Big) \wedge \operatorname{Re}(xyd\overline{x}) + \frac{1}{|x|^2} \operatorname{Re}(dx.y \wedge d\overline{x}),$$
$$\Xi_2 = \frac{1}{|x|^2} \operatorname{Re}(xdy \wedge d\overline{x}).$$

One has $d\xi = \Xi_1 + \Xi_2$ *and* $(d\xi)^3 = \Xi_2^3$.

Proof of Lemma A: The decomposition $d\xi = \Xi_1 + \Xi_2$ is obvious. Since $(d\xi)^3 = \Xi_2^3 + 3\Xi_2^2\Xi_1$, it suffices to prove that $\Xi_2^2\Xi_1 = 0$ to conclude. Let $G_4(\xi) = \{A \in \mathrm{SO}(4) : f^*\xi = \xi \text{ where } f(x,y) = (Ax, y)\}$. It is easily seen that $G_4(\xi) = \mathrm{Sp}(1) = \mathbb{S}^3$. Thus, one can work out computations at points of the form $(x, y) = (x_0, 0, 0, 0, y_0, y_1, y_2, y_3)$. One gets

$$\Xi_1 = \frac{2}{x_0^2}(-y_1 dx_2 \wedge dx_3 + y_2 dx_1 \wedge dx_3 - y_3 dx_1 \wedge dx_2),$$
$$\Xi_2 = \frac{1}{x_0} \sum_{i=0}^{3} dy_i \wedge dx_i.$$

A direct computation shows that

$$\Xi_2^2\Xi_1 = 4dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_0 \wedge (y_1 dy_1 + y_2 dy_2 + y_3 dy_3),$$

and since $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$, it follows that $\Xi_2^2\Xi_1 = 0$. ∎

Remark A: Let $\Delta_4(\xi) = \{A \in \mathrm{SO}(4) : f^*\xi = \xi \text{ where } f(x,y) = (Ax, Ay)\}$. Since $\Delta_4(\xi) = \{\phi_x \colon y \mapsto xyx^{-1} \text{ where } x \in \mathbb{S}^3\} = \mathrm{SO}(\mathrm{Im}\,\mathbb{H}) \simeq \mathrm{SO}(3)$, actually one could reduce the computations of Lemma A starting with a point of the form $(x,y) = (x_0, 0, 0, 0, y_0, y_1, 0, 0)$.

3.2 SYMPLECTIC SURGERY. Using the symplectic tubular neighbourhood theorem of A. Weinstein, one gets that any Lagrangian embedding $j \colon \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow \mathbb{C}^4$ can be extended to a symplectic diffeomorphism (still denoted by j) of a tubular neighbourhood of the zero section of $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$ into a tubular neighbourhood of $j(\mathbb{S}^1 \times \mathbb{S}^3)$ in $\mathbb{C}^4$. Let $N_0^8 = \{|y| \le \epsilon\}$ denote a closed tubular neighbourhood of the zero section of $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$ and let $N^8 = j(N_0^8)$. We also set $\Sigma_0^7 = \partial N_0^8 = \{|y| = \epsilon\} \simeq \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3$, $\Sigma^7 = j(\Sigma_0^7)$ and $K^8 = \mathbb{C}^4 \setminus \mathrm{Int}(N^8)$. Let f_n be the diffeomorphisms of $T^1\mathbb{H}^*$ given by

$$f_n \colon \mathbb{H}^* \times \mathbb{S}^3 \longrightarrow \mathbb{H}^* \times \mathbb{S}^3$$
$$(x,y) \longmapsto (xy^n, y).$$

Each f_n gives rise to a diffeomorphism of Σ_0^7 (still denoted by f_n). Moreover, for every $n \neq m$ the maps f_n and f_m belong to different arcwise connected components of $\mathrm{Diff}(\mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3)$, because they induce different maps in homology. Let V_n^8 be the manifold obtained by gluing K^8 and N^8 along their boundaries via the diffeomorphism $\tilde{f}_n = j \circ f_n \circ j^{-1}$:

$$V_n^8 = K^8 \cup_{\tilde{f}_n \colon \Sigma^7 \longrightarrow \Sigma^7} N^8.$$

This surgery can be made symplectic. Indeed, if $Z = (X, Y)$ is a tangent vector at $T_{(x,y)}\Sigma_0^7 \simeq T_x(\mathbb{S}^1 \times \mathbb{S}^3) \times T_y(\mathbb{S}^3)$, a straightforward computation shows that $(f_n^*\xi)_{(x,y)}(Z) = \xi_{(x,y)}(Z) + n < y^2, Y >$.

LEMMA B: *The forms $f_n^*\xi$ and ξ are homotopic through contact forms.*

Proof of Lemma B: Let $\alpha_{(x,y)}(Z) = < y^2, Y >$ and let $\xi_t = \xi + tn\alpha$. The form α is exact, since

$$\alpha = < y^2, dy > = < 1, \frac{dy}{y^2} > = - < 1, d(y^{-1}) > = - < 1, d\bar{y} > = -dy_0.$$

Thus $(d\xi_t)^3 = (d\xi)^3$ and, by Lemma A, $(d\xi)^3 = \Xi_2^3$. Moreover, we have $\alpha \wedge \Xi_2^3 = 0$, so

$$\xi_t \wedge (d\xi_t)^3 = (\xi + tn\alpha) \wedge \Xi_2^3 = \xi \wedge \Xi_2^3 = \xi \wedge (d\xi)^3.$$

For each t, ξ_t is a contact form. ∎

By the theorem of J. W. Gray (cf. [6]), we then deduce that $f_n^*\xi$ and ξ are isotopic. Therefore, the manifold V_n^8 admits a symplectic structure which coincides outside of a large ball with the standard symplectic structure of $\mathbb{C}^4$. Moreover, it is easy to check that $H_2(V_n^8, \mathbb{Z}) = 0$. It then follows that V_n^8 is diffeomorphic to $\mathbb{C}^4$ from a result of Y. Eliashberg, A. Floer, M. Gromov and D. McDuff (see [11], p. 653 theorem 1.5 and [4] for a detailed account of their personal contributions).

3.3 Proof of Theorem 1 for $n = 4$. We mimic what is done in [5]. The idea is to show that $V_n^8 \simeq \mathbb{C}^4$ implies $\sigma_3(j, v) = 0$. We first define explicit generators of each homology group involved in the proof. Let

$$\begin{aligned} H_3(\mathbb{S}^3 \times \mathbb{S}^1) &= \mathbb{Z} < \alpha_0 >, & H_3(K^8) &= \mathbb{Z} < H >, \\ H_3(\Sigma_0^7) &= \mathbb{Z} < \alpha > \oplus\, \mathbb{Z} < \eta >, & H_3(N^8) &= \mathbb{Z} < A >, \\ H_3(\Sigma^7) &= \mathbb{Z} < a > \oplus\, \mathbb{Z} < h > . \end{aligned}$$

Here $\mathbb{Z} < g >$ denotes the infinite cyclic group generated by g, α_0 is the 3-cycle $\{1\} \times [\mathbb{S}^3]$ of $\mathbb{S}^3 \times \mathbb{S}^1$, α is $\{1\} \times [\mathbb{S}^3] \times \{1\}$ in Σ_0^7, η is $\{1\} \times \{1\} \times [\mathbb{S}^3]$, $a = j_*\alpha$, $h = j_*\eta$, $A = j_*\alpha_0$ and H is such that $lk(j(\mathbb{S}^1 \times \mathbb{S}^3), H) = 1$. Let $i\colon \Sigma^7 \longrightarrow K^8$ and $\iota\colon \Sigma^7 \longrightarrow N^8$ denote the natural inclusions and let k be the composition $\iota \circ \tilde{f}_n$. The maps i and k induce two homomorphisms:

$$i_*\colon H_3(\Sigma^7) \longrightarrow H_3(K^8) \quad \text{and} \quad k_*\colon H_3(\Sigma^7) \longrightarrow H_3(N^8).$$

It is obvious that

$$\begin{aligned} k_*(h) &= nA, k_*(a) = A, \\ i_*(h) &= H, i_*(a) = lk(j(\mathbb{S}^1 \times \mathbb{S}^3), a). \end{aligned}$$

By the choice of the unitary tangent field v on $\mathbb{S}^1 \times \mathbb{S}^3$, we get $\alpha = Jv(\alpha_0)$ (with obvious notation). Since j is a symplectomorphism, it is easy to see that

$$lk(j(\mathbb{S}^1 \times \mathbb{S}^3), a) = lk(j(\mathbb{S}^1 \times \mathbb{S}^3), Jv(\{1\} \times \mathbb{S}^3)) = < \sigma_3(j, v), \alpha_0 > .$$

Hence, with respect to the basis (a, h) and (A, H), the matrix of the map $i_* \oplus (-k_*)$ is given by

$$M = \begin{pmatrix} -1 & -n \\ < \sigma_3(j, v), \alpha_0 > & 1 \end{pmatrix}.$$

On the other hand, from the Mayer–Vietoris sequence

$$0 \simeq H_4(V_n^8) \longrightarrow H_3(\Sigma^7) \overset{i_*\oplus(-k_*)}{\longrightarrow} H_3(K^8) \oplus H_3(N^8) \longrightarrow H_3(V_n^8) \simeq 0$$

one deduces that the map $i_* \oplus (-k_*)$ is an isomorphism. The determinant of the matrix M must be ± 1. Since this determinant is equal to $-1+n < \sigma_3(j,v), \alpha_0 >$, it follows that $< \sigma_3(j,v), \alpha_0 >= 0$. ∎

3.4 PROOF OF THEOREM 1 FOR $n = 8$. It suffices to consider the Cayley algebra $\mathbb{O}$ instead of $\mathbb{H}$ and to mimic the above proof. Nevertheless, some points of this proof deserve more explanations since the nonassociativity of $\mathbb{O}$ introduces some new difficulties.

Recall that the associator $[x,y,z]$ of a triple of elements of $\mathbb{O}$ is the difference $(xy)z - x(yz)$. In $\mathbb{O}$, a weak form of associativity holds, namely, the associator is a trilinear map. Moreover, $[y, y^{-1}, z] = 0$. We need this property to prove that the map

$$\begin{aligned} f_1\colon \mathbb{O}^* \times \mathbb{S}^7 &\longrightarrow \mathbb{O}^* \times \mathbb{S}^7 \\ (x,y) &\longmapsto (xy,y) \end{aligned}$$

is a diffeomorphism. Indeed,

$$\begin{aligned} x_1 y = x_2 y &\Longrightarrow (x_1 y)y^{-1} = (x_2 y)y^{-1} \\ &\Longrightarrow x_1(yy^{-1}) = x_2(yy^{-1}), \end{aligned}$$

since $[y, y^{-1}, z] = 0$. As $f_n = f_1 \circ \cdots \circ f_1$, f_n is also a diffeomorphism.

Difficulties arise when one wants to establish an analogue of Lemma A. This is due to the fact that the group

$$G_8(\xi) = \{A \in \mathrm{SO}(8)\colon f^*\xi = \xi \text{ where } f(x,y) = (Ax, y)\}$$

is very small: it contains only $\pm\,\mathrm{Id}$! (Here $\xi_{(x,y)} = \frac{1}{|x|^2}\mathrm{Re}((xy)d\bar{x})$, with $(x,y) \in \mathbb{O}^* \times \mathbb{S}^7$.)

LEMMA C: *Let*

$$\begin{aligned} \Xi_1 =& d\Big(\frac{1}{|x|^2}\Big) \wedge \mathrm{Re}(xyd\bar{x}) + \frac{1}{|x|^2}\mathrm{Re}(dx.y \wedge d\bar{x}), \\ \Xi_2 =& \frac{1}{|x|^2}\mathrm{Re}(xdy \wedge d\bar{x}). \end{aligned}$$

One has $d\xi = \Xi_1 + \Xi_2$ *and* $(d\xi)^7 = \Xi_2^7$.

Proof of Lemma C: Similarly to Lemma A, the nontrivial step is to prove that $\Xi_2^6\Xi_1 = 0$. Let $\Delta_8(\xi) = \{A \in \mathrm{SO}(8)\colon f^*\xi = \xi \text{ where } f(x,y) = (Ax, Ay)\}$. It turns out that $\Delta_8(\xi)$ is the exceptional Lie group G_2 (see [9] p. 114). This Lie group can be seen as a Lie subgroup of $\mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \simeq \mathrm{SO}(7)$. Moreover, it is well known that $G_2/\mathrm{SU}(3) \simeq \mathbb{S}^6$. Thus, one can work out computations at points (x,y) with $x = (x_0, x_1, 0, 0, 0, 0, 0, 0)$. Let Vol_x denote $dx_0 \wedge \cdots \wedge dx_7$ and let $\mathrm{Vol}_{i,j,k}$ denote the form

$$dy_0 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge \widehat{dy_k} \wedge \cdots \wedge dy_7$$

where dy_i, dy_j and dy_k are missing and set

$$\beta_{i,j,k} = \mathrm{Vol}_{i,j,k} \wedge (y_i dy_i + y_j dy_j + y_k dy_k).$$

A direct computation shows

$$\begin{aligned}\Xi_2^6 \Xi_1 = 1440\Big(&\frac{1}{(x_0^2 + x_1^2)^4} \mathrm{Vol}_x \wedge [\beta_{1,2,3} - \beta_{1,4,5} - \beta_{1,6,7}] \\ &+ \frac{x_0^2 - x_1^2}{(x_0^2 + x_1^2)^5} \mathrm{Vol}_x \wedge [\beta_{2,5,7} - \beta_{2,4,6} - \beta_{3,5,6} - \beta_{3,4,7}] \\ &+ \frac{2x_0 x_1}{(x_0^2 + x_1^2)^5} \mathrm{Vol}_x \wedge [\beta_{2,5,6} + \beta_{2,4,7} + \beta_{3,5,7} - \beta_{3,4,6}]\Big).\end{aligned}$$

Then, since $\sum_{i=0}^{7} y_i^2 = 1$, all the $\beta_{i,j,k}$ vanish. ∎

The rest of the proof of Theorem 1 runs exactly as in 3.3.

4. Some generalizations

4.1 On Lagrangian embeddings of some quotients of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Let Γ be a finite multiplicative subgroup of $\mathbb{S}^3$. The quotient $\mathbb{S}^3/\Gamma$ is an orientable Riemannian homogeneous 3-dimensional manifold of positive constant curvature (every finite subgroup of $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ is fixed point free, see [14]). Let j be a Lagrangian embedding of $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ into $\mathbb{C}^4$. As $H^3(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma) \simeq \mathbb{Z} \oplus \Gamma'$ where $\Gamma' = \Gamma/[\Gamma, \Gamma]$ is the abelianization of Γ, the linking class $\sigma_3(j,v)$ splits in a linear part and a torsion one.

Proposition: (1) *Let $j\colon \mathbb{S}^1 \times \mathbb{S}^3/\Gamma \longrightarrow \mathbb{C}^4$ be any Lagrangian embedding and let v be a nonvanishing vector field on $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ tangent to the factor $\mathbb{S}^1$. Then, the linear part of $\sigma_3(j,v)$ vanishes.*
(2) *Let $j\colon \mathbb{S}^1 \times \mathbb{R}P^7 \longrightarrow \mathbb{C}^8$ be any Lagrangian embedding and let v be a nonvanishing vector field on $\mathbb{S}^1 \times \mathbb{R}P^7$ tangent to the factor $\mathbb{S}^1$. Then, the linear part of $\sigma_7(j,v)$ vanishes.*

Remark B: If M^n admits a Lagrangian immersion in $\mathbb{C}^n$, then $\mathbb{S}^1 \times M^n$ admits a Lagrangian embedding into $\mathbb{C}^{n+1}$ (see [1], for instance). As any oriented 3-dimensional manifold is parallelizable, it follows from the Gromov–Lees theorem that such 3-manifolds admit a Lagrangian immersion. Thus $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ admits a Lagrangian embedding. Similarly, since $\mathbb{R}P^7$ is parallelizable, it admits a Lagrangian immersion and $\mathbb{S}^1 \times \mathbb{R}P^7$ admits a Lagrangian embedding.

Remark C: There is only one (proper) finite multiplicative subgroup Γ of $\mathbb{S}^3 \subset \mathbb{H}$ such that $\Gamma' = 0$ (and thus σ_3 has no torsion part). This subgroup is the lift of the dodecahedral subgroup of SO(3) to the double covering $\mathbb{S}^3 \longrightarrow \mathrm{SO}(3)$. The quotient $\Sigma^3 = \mathbb{S}^3/\Gamma$ is the Poincaré sphere.

4.2 Proof of the proposition.

Proof of (1): We first assume that Γ acts by left multiplication on $\mathbb{S}^3$. The diffeomorphism

$$f_n\colon \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3$$
$$(x, y) \longmapsto (xy^n, y)$$

gives a diffeomorphism on the quotient $\mathbb{S}^1 \times (\mathbb{S}^3/\Gamma) \times \mathbb{S}^3$. Since $G_4(\xi) = \mathbb{S}^3$, the contact structure ξ on $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$ also induces a contact structure on $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma)$, still denoted by ξ. The pull-back of ξ by f_n is

$$(f_n^*\xi)_{(x,y)}(Z) = \xi_{(x,y)}(Z) + n < y^2, Y >,$$

where Z is a tangent vector at a point (x, y) of $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma)$. As in section 3.2, one can perform a symplectic surgery on $\mathbb{C}^4$ and denote the resulting manifold by V_n^8. It is easy to check that $H_2(V_n^8) \simeq 0$ and thus V_n^8 is diffeomorphic to $\mathbb{C}^4$. Repeating the arguments of Theorem 1, one gets

$$< \sigma_3(j, v), \alpha_0 >= 0,$$

where α_0 is the 3-cycle of $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ represented by $\{1\} \times [\mathbb{S}^3/\Gamma]$. Note that the third cohomology group of $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ has some torsion, namely, $H^3(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma, \mathbb{Z}) \simeq \mathbb{Z} \oplus \Gamma'$. Since Γ is finite, Γ' is a torsion group.

If the group Γ acts by right multiplication, one has to consider the contact form $\xi_{(x,y)} = \frac{1}{|x|^2} \mathrm{Re}(yx d\bar{x})$ and the diffeomorphisms $(x, y) \longmapsto (y^n x, y)$.

Proof of (2): The proof is similar to the proof of point (1). But since $G_8(\xi) \simeq \mathbb{Z}_2$, it only applies to the quotient $\mathbb{S}^7/\mathbb{Z}_2 = \mathbb{R}P^7$. ∎

5. Isotopy classes of embeddings

5.1 General theory of A. Haefliger and M. Hirsch [8]. Let M^n be a n-dimensional ($n \geq 4$) compact connected orientable manifold without boundary and let $j\colon M^n \longrightarrow \mathbb{C}^n \simeq \mathbb{R}^{2n}$ be an embedding. Let M_0^n denote the manifold $M^n \setminus \{x_0\}$ where x_0 is any point of M^n. Let $D_1 \subset D_2$ be two embedded open disks centered at x_0 and let $\lambda\colon M^n \longrightarrow [0,1]$ be a C^∞ map such that $\lambda \equiv 1$ on $M^n \setminus D_2$ and $\lambda \equiv 0$ on D_1. Given a nonvanishing normal vector field ν on M_0, let us set

$$j_\nu(x) = \begin{cases} j(x) & \text{if } x \in D_1, \\ j(x) + \lambda(x)\nu_x & \text{if } x \in M^n \setminus D_1. \end{cases}$$

Finally, let $X = \mathbb{C}^n \setminus j(M^n \setminus D_2)$. There exists a unique (up to homotopy) normal vector field ν on M_0^n such that

$$[j_\nu(M^n)] = 0 \quad \text{in } H_n(X, \mathbb{Z}) \simeq H^{n-1}(M^n \setminus D_2, \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

We call this vector field the **associated vector field** $\nu(j)$ to the embedding j. If ν and ν' are two nonvanishing normal vector fields, the following simple relation holds:

$$[j_{\nu'}(M^n)] - [j_\nu(M^n)] = d(\nu', \nu)$$

where $d(\nu', \nu)$ is the difference class, the first obstruction to homotopy between ν' to ν.

Let $T_{2n,n+1}(M_0^n)$ be the associated bundle to the frame bundle of M_0^n whose fiber is the Stiefel manifold $V_{2n,n+1}$ of $(n+1)$-frames of $\mathbb{R}^{2n}$, with $\mathrm{Gl}(n, \mathbb{R})$ acting in a natural way on the first n vectors of a frame. Every embedding j gives rise to a natural section σ_j of $T_{2n,n+1}(M_0^n)$ given by

$$\begin{aligned} \sigma_j\colon M_0^n &\longrightarrow T_{2n,n+1}(M_0^n) \\ x &\longmapsto [(dj_x(\mathcal{R}), \nu_x(j))], \end{aligned}$$

where $\mathcal{R}$ is any frame in $T_x M_0^n$. Let $\Gamma(T_{2n,n+1}(M_0^n))$ be the space of (continuous) sections of $T_{2n,n+1}(M_0^n)$ and let $E(M^n, \mathbb{R}^{2n})$ be the space of embeddings of M^n into $\mathbb{R}^{2n}$.

Theorem (A. Haefliger, M. Hirsch): *The map $j \longrightarrow \sigma_j$ induces a bijection on the π_0-level between the space of embeddings $E(M^n, \mathbb{R}^{2n})$ and the space of sections $\Gamma(T_{2n,n+1}(M_0^n))$.*

There is a 1-1 correspondence between $\pi_0(\Gamma(T_{2n,n+1}(M_0^n)))$ and the cohomology group $H^{n-1}(M_0^n, \pi_{n-1}(V_{2n,n+1}))$. Thus $\pi_0(E(M^n, \mathbb{R}^{2n}))$ is in bijection with $H^{n-1}(M^n, \mathbb{Z}_2)$ if n is even, and with $H^{n-1}(M^n, \mathbb{Z})$ if n is odd.

5.2 THE CASE $M^n = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ WITH $n = 4$ OR 8. Throughout this section n will be 4 or 8. For short, we put $E_n = E(\mathbb{S}^1 \times \mathbb{S}^{n-1}, \mathbb{R}^{2n})$. According to the preceding subsection, $\pi_0(E_n)$ is in bijection with $\mathbb{Z}_2$.

Let v be a nonvanishing vector field on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ tangent to the factor $\mathbb{S}^1$. By using the quaternionic or the octonian structure (we see $\mathbb{O}$ as $\mathbb{H} \oplus e\mathbb{H}$), we can define a global n-frame $\mathcal{R} = \{e_1, e_2, \dots, e_n\}$ on M^n as

$$\begin{aligned}(e_1, \dots, e_4) &= (v, iv, jv, kv) && \text{if } n = 4,\\ (e_1, \dots, e_8) &= (v, iv, jv, kv, ev, eiv, ejv, ekv) && \text{if } n = 8.\end{aligned}$$

The map $j \mapsto \sigma_j$ can be identified with

$$\begin{aligned}\Phi\colon E_n &\longrightarrow \Gamma(T_{2n,n+1}(M_0^n))\\ j &\longmapsto \sigma_j\colon x \mapsto (dj_x(\mathcal{R}), \nu_x(j)).\end{aligned}$$

The n-frame $\mathcal{R}$ gives a trivialization of $T_{2n,n+1}(M_0^n)$. There are obvious identifications

$$\begin{aligned}\pi_0(\Gamma(T_{2n,n+1}(M_0^n))) &\simeq [M_0^n, V_{2n,n+1}] \simeq [\mathbb{S}^1 \vee \mathbb{S}^{n-1}, V_{2n,n+1}]\\ &\simeq [\mathbb{S}^1, V_{2n,n+1}] \times [\mathbb{S}^{n-1}, V_{2n,n+1}]\\ &\simeq \pi_{n-1}(V_{2n,n+1}).\end{aligned}$$

As usual $\mathbb{S}^1 \vee \mathbb{S}^{n-1}$ denotes the one-point union of $\mathbb{S}^1$ and $\mathbb{S}^{n-1}$. Let θ be any point of $\mathbb{S}^1$. The map $\Phi_\sharp\colon \pi_0(E_n) \longrightarrow \mathbb{Z}_2$ is just given by the class of the map

$$\begin{aligned}\{\theta\} \times \mathbb{S}^{n-1} &\longrightarrow V_{2n,n+1}\\ x &\longmapsto (dj_x(\mathcal{R}), \nu_x(j))\end{aligned}$$

in $\pi_{n-1}(V_{2n,n+1}) \simeq \mathbb{Z}_2$. We call this class $\epsilon_{\mathcal{R}}(j)$. This modulo 2 number tells us to which component of E_n the map j belongs.

5.3 PROOF OF THEOREM 2. Here again, n is 4 or 8. If j is Lagrangian, it is natural to introduce another map

$$\begin{aligned}\overline{T}_j\colon \{\theta\} \times \mathbb{S}^{n-1} &\longrightarrow \mathrm{SO}(2n)\\ x &\longmapsto (dj_x(\mathcal{R}), Jdj_x(\mathcal{R})).\end{aligned}$$

Let F be the forgetting map $\mathrm{SO}(2n) \longrightarrow V_{2n,n+1}$. The composition yields a new map

$$\begin{aligned}F \circ \overline{T}_j\colon \{\theta\} \times \mathbb{S}^{n-1} &\longrightarrow V_{2n,n+1}\\ x &\longmapsto (dj_x(\mathcal{R}), Jdj_x(v)).\end{aligned}$$

By Theorem 1, the nonvanishing vector field $Jdj(v)$ is exactly $\nu(j)$ (on M_0^n). Thus, the class $[F \circ \overline{T}_j]$ of this map in $\pi_{n-1}(V_{2n,n+1})$ is $\epsilon_{\mathcal{R}}(j)$. The map F induces the modulo 2 reduction at the π_{n-1}-level. Indeed

$$\begin{array}{ccccc} \pi_{n-1}(\mathrm{SO}(2n)) & \xrightarrow{F_\#} & \pi_{n-1}(V_{2n,n+1}) & \longrightarrow & \pi_{n-2}(\mathrm{SO}(n-1)) \\ \| & & \| & & \| \\ \mathbb{Z} & \xrightarrow{\bmod 2} & \mathbb{Z}_2 & \longrightarrow & 0. \end{array}$$

The relation: $[F \circ \overline{T}_j] = F_\sharp \cdot [\overline{T}_j]$ implies that $\epsilon_{\mathcal{R}}(j)$ is the modulo 2 reduction of the class $[\overline{T}_j] \in \pi_{n-1}(\mathrm{SO}(2n))$. Let $i\colon U(n) \longrightarrow \mathrm{SO}(2n)$ be the natural inclusion; by construction the map $\overline{T}_j$ factors through $U(n)$, i.e. $\overline{T}_j = i \circ T_j$ for some $T_j\colon \{\theta\} \times \mathbb{S}^{n-1} \longrightarrow U(n)$.

If $n = 8$: The study of the long homotopy exact sequence associated to the fibration $U(8) \longrightarrow \mathrm{SO}(16) \longrightarrow \mathrm{SO}(16)/U(8)$ shows that the map $i_\sharp\colon \mathbb{Z} = \pi_7(U(8)) \longrightarrow \pi_7(\mathrm{SO}(16)) = \mathbb{Z}$ induced by i is the multiplication by 2. Thus, the modulo 2 reduction of the class $[\overline{T}_j]$ in $\pi_7(\mathrm{SO}(16))$ is zero, i.e. $\epsilon_{\mathcal{R}}(j) = 0$.

If $n = 4$: Again by the use of the homotopy sequence of the fibration $U(4) \longrightarrow \mathrm{SO}(8) \longrightarrow \mathrm{SO}(8)/U(4)$, one shows that the map $i_\sharp\colon \mathbb{Z} = \pi_3(U(4)) \longrightarrow \pi_3(\mathrm{SO}(8)) = \mathbb{Z}$ is an isomorphism. Thus, *a priori*, $\epsilon_{\mathcal{R}}(j)$ can be any of the two elements of $\mathbb{Z}_2$. Let us see that both values are actually taken.

Let $\mathbb{S}^3$ be the unit sphere of $\mathbb{H}$, v the outward unit normal field of $\mathbb{S}^3$ and R the 3-frame (iv, jv, kv). Any Lagrangian immersion g of $\mathbb{S}^3$ into $\mathbb{C}^3$ gives rise to a map

$$\begin{aligned} dg^{\mathbb{C}}\colon \mathbb{S}^3 &\longrightarrow U(3) \subset \mathrm{SO}(6) \\ x &\longmapsto (dg_x(R), Jdg_x(R)). \end{aligned}$$

Let g_0 and g_1 be two Lagrangian immersions of $\mathbb{S}^3$ into $\mathbb{C}^3$ such that $[dg_0^{\mathbb{C}}]$ is zero in $\pi_3(U(3))$ and $[dg_1^{\mathbb{C}}]$ is a generator of $\pi_3(U(3))$. Then $f_i = h \times g_i$, $(i = 0, 1)$ are two Lagrangian immersions of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C} \times \mathbb{C}^3$ (h denotes the natural (Lagrangian) inclusion $\mathbb{S}^1 \subset \mathbb{C}$). Of course $df_i^{\mathbb{C}} = dh^{\mathbb{C}} \times dg_i^{\mathbb{C}}$. A result of [1] claims that there exist two Lagrangian embeddings j_0 and j_1 of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C}^4$ such that $dj_i^{\mathbb{C}}$ is homotopic to $df_i^{\mathbb{C}}$ (as maps from $\mathbb{S}^1 \times \mathbb{S}^3$ to $U(4)$). Thus the maps T_{j_i} are homotopic to T_{f_i}. Let θ be a point of $\mathbb{S}^1$ and l be the inclusion:

$$\begin{aligned} l\colon U(3) &\longrightarrow U(4) \\ A &\longmapsto \begin{pmatrix} dh_\theta^{\mathbb{C}}(v) & 0 \\ 0 & A \end{pmatrix}. \end{aligned}$$

Then $T_{f_i} = l \circ dg_i^{\mathbb{C}}$. As l induces an isomorphism at the π_3-level, T_{f_i} represents a generator of $\pi_3(U(4))$ if $i = 1$ and the zero class if $i = 0$. Therefore, $\epsilon_{\mathcal{R}}(j_i) = i$.

5.4 THE OBSTRUCTIONS $\Delta(\cdot,\cdot)$ AND $\epsilon(\cdot,\cdot)$. We first need a preparatory lemma. Let $n \geq 2$ and let Ψ be the classical isomorphism between $U(n)$ and $\mathbb{S}^1 \times \mathrm{SU}(n)$. We denote by $\iota_{\mathrm{SU}(n)} \in H^3(\mathrm{SU}(n), \mathbb{Z})$ the characteristic class of $\mathrm{SU}(n)$ and by a the class $\Psi^* \iota_{\mathrm{SU}(n)}$. The following lemma is well known (see theorem 7.16 p. 146 of [13]).

LEMMA D: *The "degree map"* $[M^n, U(n)] \xrightarrow{deg} H^3(M^n, \mathbb{Z})$, $f \longmapsto f^*a$ *is a group homomorphism.*

Let M^n be a manifold admitting a Lagrangian immersion into $\mathbb{C}^n$, s_0 a unitary trivialization of $TM^n \otimes \mathbb{C}$ and let $(e_1, \ldots, e_n)$ be an unitary basis of $\mathbb{C}^n$. The theorem of Gromov–Lees asserts that the map

$$\begin{aligned} I_{Lag}(M^n, \mathbb{C}^n) &\longrightarrow C^0(M^n, U(n)) \\ j &\longmapsto dj^{\mathbb{C}}(s_0) \end{aligned}$$

is a weak homotopy equivalence. By using this theorem, it is easy to build invariants for Lagrangian immersions. For instance, the class $\delta(j, s_0) = (dj^{\mathbb{C}}(s_0))^* a$ is an obvious one. (Lemma D shows that $\delta(j, s_0)$ does not depend on the choice of the unitary basis of $\mathbb{C}^n$.) If s_1 is another trivialization, then there exists a map $f\colon M^n \longrightarrow U(n)$ such that $s_1 = s_0 f$. By Lemma D

$$\delta(j, s_1) = (dj^{\mathbb{C}}(s_1))^* a = (dj^{\mathbb{C}}(s_0) \cdot f)^* a = \delta(j, s_0) + f^* a.$$

Therefore, if j_0 and j_1 are two Lagrangian immersions, the class

$$\Delta(j_0, j_1) = \delta(j_1, s_0) - \delta(j_0, s_0)$$

does not depend on the choice of the trivialization. By construction, this class is an obstruction to the existence of a Lagrangian regular homotopy between j_0 and j_1. Let $\epsilon(j_0, j_1)$ be the modulo 2 reduction of $\Delta(j_0, j_1)$.

LEMMA E: *Let j_0, j_1 be two Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C}^4$. Then $\epsilon(j_0, j_1)$ is the only obstruction to the existence of a smooth isotopy between j_0 and j_1.*

Proof of Lemma E: Let k be the inclusion $\{0\} \times \mathbb{S}^3 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^3$. One has the following commutative diagram:

$$\begin{array}{ccccc} \pi_3(U(4)) \simeq [\mathbb{S}^3, U(4)] & \xrightarrow{\deg} & H^3(\mathbb{S}^3, \mathbb{Z}) & \xrightarrow{(k^*)^{-1}} & H^3(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{Z}) \\ \Big\downarrow F\circ i & & \Big\downarrow \rho_2 & & \Big\downarrow \rho_2 \\ \pi_3(V_{8,5}) \simeq [\mathbb{S}^3, V_{8,5}] & \xrightarrow{\deg} & H^3(\mathbb{S}^3, \mathbb{Z}_2) & \xrightarrow{(k^*)^{-1}} & H^3(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{Z}_2) \end{array}$$

where k^* and the degree maps are isomorphisms. (The second degree map is just $f \mapsto f^*\iota_{V_{8,5}}$ where $\iota_{V_{8,5}} \in H^3(V_{8,5}, \mathbb{Z}_2)$ is the characteristic class of $V_{8,5}$.) The first vertical maps between the cohomology groups is induced by the coefficient homomorphism

$$\mathbb{Z} = \pi_3(U(4)) \xrightarrow{\text{mod } 2} \pi_3(V_{8,5}) = \mathbb{Z}_2$$

(see, for instance, [2] pp. 275–278 for this point). Let s_0 be the trivialization of $TM^n \otimes \mathbb{C}$ induced by the global n-frame $\mathcal{R}$. It is readily seen that: $(k^*)^{-1} \deg[T_j] = \delta(j, s_0)$ and that $(k^*)^{-1} \deg \epsilon_{\mathcal{R}} = (k^*)^{-1} \deg[F \circ i \circ T_j] = \rho_2 \delta(j, s_0)$. Hence

$$\epsilon(j_0, j_1) = (k^*)^{-1} \deg(\epsilon_{\mathcal{R}}(j_0) - \epsilon_{\mathcal{R}}(j_1)). \qquad \blacksquare$$

5.5 The action of $\mathrm{Diff}(M^n)$ on $\pi_0(E(M^n, \mathbb{C}^n))$. We first recall an alternative description of the solution to the embeddings classification problem. Any embedding $j\colon M^n \longrightarrow \mathbb{R}^{2n}$ gives rise to a $\mathbb{Z}_2$-equivariant map

$$M^n \times M^n - \Delta M^n \longrightarrow \mathbb{S}^{2n-1}$$
$$(x, y) \longmapsto \frac{j(x) - j(y)}{|j(x) - j(y)|}$$

(ΔM^n denotes the diagonal of $M^n \times M^n$). Let $\mathcal{S}$ be the space of $\mathbb{Z}_2$-equivariant maps of $M^n \times M^n - \Delta M^n$ into $\mathbb{S}^{2n-1}$. A celebrated result of A. Haefliger [7] states that, if $n \geq 4$, there is a one-to-one correspondence between $\pi_0(E(M^n, \mathbb{R}^{2n}))$ and the set $\pi_0(\mathcal{S})$. The computation of $\pi_0(\mathcal{S})$ is a classical problem in homotopy theory. Let $M^* = (M^n \times M^n - \Delta M^n)/\mathbb{Z}_2$ be the reduced symmetric product of M^n and P be the bundle $(M^n \times M^n - \Delta M^n) \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow M^*$. There is a natural bijection between $\mathcal{S}$ and the space of sections of P, thus $\pi_0(\mathcal{S}) = \pi_0(\Gamma(P))$. Since the fiber of P is the sphere $\mathbb{S}^{2n-1}$, one has $\pi_0(E(M^n, \mathbb{R}^{2n})) \simeq \pi_0(\Gamma(P)) \simeq H^{2n-1}(M^*, \mathbb{Z})$.

Lemma G: *The action of* $\mathrm{Diff}(\mathbb{S}^1 \times \mathbb{S}^3)$ *on* $\pi_0(E(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{C}^4))$ *is trivial.*

Proof of Lemma G: A diffeomorphism ϕ of M^n induces a $\mathbb{Z}_2$-equivariant diffeomorphism of $M^n \times M^n - \Delta M^n$ and thus a diffeomorphism Φ of M^*. The action of ϕ on $\pi_0(E(M^n, \mathbb{R}^{2n}))$ is given by the induced isomorphism Φ^* on $H^{2n-1}(M^*, \mathbb{Z})$. It is well known that for an orientable even-dimensional manifold M^n, the cohomology group $H^{2n-1}(M^*, \mathbb{Z})$ is isomorphic to $H^{n-1}(M^n, \mathbb{Z}_2)$. In our case $M^n = \mathbb{S}^1 \times \mathbb{S}^3$ and one gets $H^7(M^*, \mathbb{Z}) = \mathbb{Z}_2$; Φ^* is simply the identity. $\blacksquare$

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